

REPORT DOCUMENTATION PAGE			Form Approved OMB NO. 0704-0188		
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1. REPORT DATE (DD-MM-YYYY) 17-08-2012		2. REPORT TYPE Conference Proceeding		3. DATES COVERED (From - To) -	
4. TITLE AND SUBTITLE Convergence of a PI Coordination Protocol in Networks with Switching Topology and Quantized Measurements			5a. CONTRACT NUMBER W911NF-10-1-0044		
			5b. GRANT NUMBER		
			5c. PROGRAM ELEMENT NUMBER 611102		
6. AUTHORS Ronald Choe, Enric Xargay , Naira Hovakimyan, Isaac Kaminer			5d. PROJECT NUMBER		
			5e. TASK NUMBER		
			5f. WORK UNIT NUMBER		
7. PERFORMING ORGANIZATION NAMES AND ADDRESSES University of Illinois - Urbana Grants and Contracts Office 109 Coble Hall Champaign, IL 61820 -6242			8. PERFORMING ORGANIZATION REPORT NUMBER		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) U.S. Army Research Office P.O. Box 12211 Research Triangle Park, NC 27709-2211			10. SPONSOR/MONITOR'S ACRONYM(S) ARO		
			11. SPONSOR/MONITOR'S REPORT NUMBER(S) 55839-NS.19		
12. DISTRIBUTION AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.					
13. SUPPLEMENTARY NOTES The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.					
14. ABSTRACT Abstract—This paper analyzes the convergence properties of a distributed proportional-integral protocol for coordination of a network of agents with multiple leaders, dynamic information flow, and quantized measurements. We show that the integral term of the protocol allows the follower agents to ‘learn’ the					
15. SUBJECT TERMS Switching topology, quantized system, cooperative control, multi-agent systems					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT UU	15. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON Naira Hovakimyan
a. REPORT UU	b. ABSTRACT UU	c. THIS PAGE UU			19b. TELEPHONE NUMBER 217-244-1672

## **Report Title**

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**Conference Name:** IEEE 51st Conference on Decision and Control

**Conference Date:** December 10, 2012



# Convergence of a PI Coordination Protocol in Networks with Switching Topology and Quantized Measurements

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**Abstract**—This paper analyzes the convergence properties of a distributed proportional-integral protocol for coordination of a network of agents with multiple leaders, dynamic information flow, and quantized measurements. We show that the integral term of the protocol allows the follower agents to ‘learn’ the reference rate, rather than have it available a priori, and also provides disturbance rejection capabilities.

## I. INTRODUCTION

Worldwide, there has been growing interest in the use of autonomous vehicles to execute complex missions without constant supervision of human operators. A key enabling element for the execution of such missions is the availability of advanced strategies for cooperative motion control of autonomous vehicles. In [1], for example, the authors address the development of robust strategies for cooperative missions in which a fleet of UAVs is required to follow collision-free paths and arrive at their respective final destinations at the same time. The distributed protocol used for group coordination, which was first introduced in [2], has a proportional-integral (PI) structure in which each agent is only required to exchange its coordination state with its neighbors, and the constant reference rate is only available to a single leader. The integral term in the consensus algorithm allows the follower UAVs to ‘learn’ the reference rate from the leader.

A generalization of this PI protocol was proposed in [3], where the authors developed an adaptive algorithm to reconstruct a time-varying reference velocity that is available only to a single leader. The paper used a passivity framework to show that a network of nonlinear agents with fixed connected topology asymptotically achieves coordination. The work in [4] also used a (discrete-time) PI protocol to synchronize networks of clocks with fixed connected information flow. In this application, the integral part of the controller was critical to eliminate the different initial clock offsets.

This paper modifies the PI protocol in [1], [2] to include multiple leaders, and analyzes the convergence properties of the protocol for coordination of a network of agents with dynamic information flow and quantized measurements, a topic that has received increased attention in recent years [5]–[10]. On one hand, the use of multiple leaders in the protocol improves robustness to a single-point failure. On the other hand, the use of finite-rate communication links and/or coarse sensors motivates the interest in quantized consensus problems. The main contribution of this paper is twofold. First, we present lower bounds on the convergence rate of the collective dynamics as a function of the number

of leaders and the *quality of service* (QoS) of the network, which in the context of this work represents a measure of the level of connectivity of the dynamic graph that captures the underlying network topology. And second, we analyze the existence of equilibria as well as the convergence properties of the collective dynamics under quantized feedback.

The paper is organized as follows. Section II describes the problem formulation. Section III presents the PI protocol adopted in this paper and analyzes its convergence properties. In Section IV, we study the collective dynamics under quantization. Simulation results are presented in Section V, while Section VI summarizes concluding remarks.

## II. PROBLEM FORMULATION

Consider a network of  $n$  integrator-agents

$$\dot{x}_i(t) = u_i(t) + d_i, \quad x_i(0) = x_{i0}, \quad i \in \mathcal{I}_n := \{1, \dots, n\}, \quad (1)$$

with dynamic information flow  $\mathcal{G}_0(t) := (\mathcal{V}_0, \mathcal{E}_0(t))$ . In the above formulation,  $x_i(t) \in \mathbb{R}$  is the coordination state of the  $i$ th agent,  $u_i(t) \in \mathbb{R}$  is its control input, and  $d_i \in \mathbb{R}$  is an unknown constant disturbance.

The control objective is to design a distributed protocol that solves the following coordination problem:

$$x_i(t) - x_j(t) \xrightarrow{t \rightarrow \infty} 0, \quad \forall i, j \in \mathcal{I}_n, \quad (2a)$$

$$\dot{x}_i(t) \xrightarrow{t \rightarrow \infty} \rho, \quad \forall i \in \mathcal{I}_n, \quad (2b)$$

where  $\rho$  is the desired (constant) reference rate.

The network and the communications between agents satisfy the following assumptions:

*Assumption 1:* The  $i$ th agent can only exchange information with a set of neighboring agents, denoted by  $\mathcal{N}_i^0(t)$ .

*Assumption 2:* Communications between two agents are bidirectional ( $\mathcal{G}_0(t)$  is undirected) and the information is transmitted continuously with no delays.

*Assumption 3:* The connectivity of  $\mathcal{G}_0(t)$  at time  $t$  satisfies the persistency of excitation (PE)-like condition

$$\frac{1}{n} \frac{1}{T} \int_t^{t+T} \mathbf{Q}_n \mathbf{L}_0(\tau) \mathbf{Q}_n^\top d\tau \geq \mu \mathbb{I}_{n-1}, \quad \forall t \geq 0, \quad (3)$$

where  $\mathbf{L}_0(t) \in \mathbb{R}^{n \times n}$  is the piecewise-constant Laplacian of the graph  $\mathcal{G}_0(t)$ , and  $\mathbf{Q}_n$  is any  $(n-1) \times n$  matrix satisfying  $\mathbf{Q}_n \mathbf{1}_n = \mathbf{0}$  and  $\mathbf{Q}_n \mathbf{Q}_n^\top = \mathbb{I}_{n-1}$ , with  $\mathbf{1}_n$  being the vector in  $\mathbb{R}^n$  whose components are all 1. The parameters  $T > 0$  and  $\mu \in [0, 1]$  characterize the QoS of the communications network, which in the context of this paper represents a measure of the level of connectivity of the dynamic graph  $\mathcal{G}_0(t)$ .

*Remark 1:* Condition (3) requires only the graph  $\mathcal{G}_0(t)$  to be connected in an integral sense, not pointwise in time. In fact, the graph may be disconnected during some interval of

Research is supported by AFOSR, ARO, and ONR.

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time or may even fail to be connected at all times. Similar type of conditions can be found in [11] and [12].

### III. DISTRIBUTED CONSENSUS PROTOCOL

#### A. Addition of Virtual Agents

The consensus protocol adopted in this paper introduces  $n_\ell$  *virtual agents* ( $1 \leq n_\ell \leq n$ ) in the network, associated with  $n_\ell$  agents. These virtual agents are implemented in  $n_\ell$  distinct agents and have the following dynamics:

$$\dot{x}_{\ell i}(t) = u_{\ell i}(t), \quad x_{\ell i}(0) = x_{\ell i0}, \quad i \in \mathcal{I}_\ell := \{1, \dots, n_\ell\},$$

where the virtual control laws  $u_{\ell i}(t)$ ,  $i \in \mathcal{I}_\ell$ , are yet to be defined. Without loss of generality, we assume that these virtual agents are implemented in agents 1 to  $n_\ell$ , that is, the  $i$ th virtual agent is implemented in the  $i$ th agent. In the context of this paper, these  $n_\ell$  agents are referred to as *leaders*, while the remaining agents are *followers*.

To limit the amount of information transmitted over the network, each leader is only allowed to exchange the state of its virtual agent with its neighbors; in other words, the  $i$ th leader can only transmit the state  $x_{\ell i}(t)$ , rather than transmitting both  $x_{\ell i}(t)$  and  $x_i(t)$ . Finally, we note that the agent and the virtual agent of a leader can exchange information uninterruptedly, as these two agents do not communicate over the network. Figure 1 presents an example illustrating the addition of two virtual agents in a network of three agents.

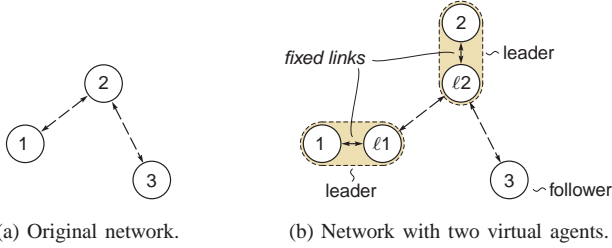


Fig. 1. Addition of  $n_\ell = 2$  virtual agents in a network of  $n = 3$  agents.

The inclusion of these  $n_\ell$  virtual agents results in a new *extended network* of  $N := n + n_\ell$  agents with a new dynamic topology  $\mathcal{G}(t)$ . According to the description above, this new topology is characterized by the following neighboring sets:

$$\begin{aligned} \mathcal{N}_i &:= \{\ell i\}, & i \in \mathcal{I}_\ell, \\ \mathcal{N}_i(t) &:= (\mathcal{N}_i^0(t) \setminus \mathcal{I}_\ell) \cup \mathcal{L}_i(t), & i \notin \mathcal{I}_\ell, \\ \mathcal{N}_{\ell i}(t) &:= (\mathcal{N}_{\ell i}^0(t) \setminus \mathcal{I}_\ell) \cup \mathcal{L}_i(t) \cup \{i\}, & i \in \mathcal{I}_\ell, \end{aligned}$$

where the vertex set  $\mathcal{L}_i(t)$  is defined as

$$\mathcal{L}_i(t) := \{\ell j : j \in (\mathcal{N}_i^0(t) \cap \mathcal{I}_\ell)\}.$$

The Laplacian  $\mathbf{L}(t)$  of the new extended graph with vertex set  $\mathcal{V} := \{\ell 1, \dots, \ell n_\ell, 1, \dots, n\}$  is given by

$$\mathbf{L}(t) = \mathbf{P}_\ell^\top \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_0(t) \end{bmatrix} \mathbf{P}_\ell + \mathbf{L}_v \in \mathbb{R}^{N \times N},$$

where  $\mathbf{P}_\ell$  is the  $(0, 1)$ -permutation matrix

$$\mathbf{P}_\ell := \begin{bmatrix} \mathbf{0} & \mathbf{I}_{n_\ell} & \mathbf{0} \\ \mathbf{I}_{n_\ell} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-n_\ell} \end{bmatrix} \in \mathbb{R}^{N \times N};$$

while  $\mathbf{L}_v$  is defined as

$$\mathbf{L}_v := \begin{bmatrix} \mathbf{I}_{n_\ell} & -\mathbf{I}_{n_\ell} & \mathbf{0} \\ -\mathbf{I}_{n_\ell} & \mathbf{I}_{n_\ell} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{N \times N}.$$

The lemma below shows that the connectivity of the graph  $\mathcal{G}(t)$  satisfies a PE-like condition similar to (3).

**Lemma 1:** Consider a network with  $n$  agents and  $n_\ell$  virtual agents, added according to the description above. If the connectivity of the original network satisfies Assumption 3, then the connectivity of the extended network verifies

$$\frac{1}{N} \frac{1}{T} \int_t^{t+T} \mathbf{Q}_N \mathbf{L}(\tau) \mathbf{Q}_N^\top d\tau \geq \mu_{n_\ell} \mathbf{I}_{N-1}, \quad \forall t \geq 0,$$

where  $\mathbf{Q}_N$  is any  $(N-1) \times N$  matrix such that  $\mathbf{Q}_N \mathbf{1}_N = \mathbf{0}$  and  $\mathbf{Q}_N \mathbf{Q}_N^\top = \mathbf{I}_{N-1}$ ; while the constant  $\mu_{n_\ell}$  characterizes the QoS of the extended network. The parameter  $\mu_{n_\ell}$  can be determined recursively from the relation

$$\mu_i = F(n + i - 1, \mu_{i-1}), \quad i = 1, \dots, n_\ell,$$

together with the initial condition  $\mu_0 = \mu$ , and

$$F(k, x_\mu) := \frac{(kx_\mu + 2) - \sqrt{(kx_\mu + 2)^2 - 4x_\mu(k+1)}}{2(k+1)}.$$

**Proof.** The proof is omitted due to space limitations.  $\square$

**Remark 2:** The concept of *virtual leader* is quite common in the consensus literature, where it is understood as an additional agent running in open loop and providing an external consensus reference state to a subgroup of follower agents; see [13]–[16] and references therein. In our framework, the *virtual agents* (or *virtual leaders*) play a different role in the network protocol and their dynamics are affected by other agents (virtual and non-virtual). From a functional perspective, the virtual agents in our framework are equivalent to the “subgroup leaders” in the work reported in [15]. The key reason for implementing these virtual agents as part of our consensus protocol is to provide a set of agents with *disturbance-free dynamics*, which –as will become clear later– allow to effectively solve the coordination problem (2).

#### B. Proportional-Integral Protocol

To solve the consensus problem (2), we adopt the protocol

$$u_{\ell i} = k_P \sum_{j \in \mathcal{N}_{\ell i}} (x_j - x_{\ell i}) + \rho, \quad i \in \mathcal{I}_\ell, \quad (4)$$

$$u_i = k_P \sum_{j \in \mathcal{N}_i} (x_j - x_i) + \chi_i, \quad i \in \mathcal{I}_n, \quad (5)$$

$$\dot{\chi}_i = k_I \sum_{j \in \mathcal{N}_i} (x_j - x_i), \quad \chi_i(0) = \chi_{i0}, \quad i \in \mathcal{I}_n, \quad (6)$$

where  $k_P > 0$  and  $k_I > 0$  are coordination gains. This protocol has a PI structure in which each agent is only required to exchange its coordination state  $x_\bullet(t)$  with its neighbors, and the reference rate  $\rho$  is only available to the  $n_\ell$  leaders. We also note that the virtual agents adjust their dynamics according to information exchanged with their neighboring agents.

The protocol (4)–(6) can be rewritten in compact form as

$$\begin{aligned} \mathbf{u}(t) &= -k_P \mathbf{L}(t) \mathbf{x}(t) + \begin{bmatrix} \rho \mathbf{1}_{n_\ell} \\ \chi(t) \end{bmatrix}, \\ \dot{\chi}(t) &= -k_I \mathbf{C}^\top \mathbf{L}(t) \mathbf{x}(t), \quad \chi(0) = \chi_0, \end{aligned} \quad (7)$$

where  $\mathbf{u}(t)$ ,  $\mathbf{x}(t)$ , and  $\chi(t)$  are defined as

$$\mathbf{u}(t) := [u_{\ell 1}(t), \dots, u_{\ell n_\ell}(t), u_1(t), \dots, u_n(t)]^\top \in \mathbb{R}^N,$$

$$\mathbf{x}(t) := [x_{\ell 1}(t), \dots, x_{\ell n_\ell}(t), x_1(t), \dots, x_n(t)]^\top \in \mathbb{R}^N,$$

$$\chi(t) := [\chi_1(t), \dots, \chi_n(t)]^\top \in \mathbb{R}^n;$$

$$\text{and } \mathbf{C}^\top := \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \end{bmatrix} \in \mathbb{R}^{n \times N}.$$

### C. Collective Dynamics and Convergence Analysis

Protocol (7) leads to the *closed-loop collective dynamics*

$$\begin{aligned}\dot{\mathbf{x}}(t) &= -k_P \mathbf{L}(t) \mathbf{x}(t) + \begin{bmatrix} \rho \mathbf{1}_{n_\ell} \\ \chi(t) + \mathbf{d} \end{bmatrix}, & \mathbf{x}(0) &= \mathbf{x}_0, \\ \dot{\chi}(t) &= -k_I \mathbf{C}^\top \mathbf{L}(t) \mathbf{x}(t), & \chi(0) &= \chi_0,\end{aligned}$$

where  $\mathbf{d} := [d_1, \dots, d_n]^\top \in \mathbb{R}^n$  is the disturbance vector. Note that the solutions (in the sense of Carathéodory [17]) of the collective dynamics above exist and are unique, since the Laplacian  $\mathbf{L}(t)$  is piecewise constant in  $t$ .

To analyze the convergence properties of the algorithm (7), we reformulate the consensus problem (2) into a stabilization problem. To this end, we define the *projection matrix*  $\mathbf{\Pi}_N$  as

$$\mathbf{\Pi}_N := \mathbf{I}_N - \frac{\mathbf{1}_N \mathbf{1}_N^\top}{N} \in \mathbb{R}^{N \times N},$$

and note that the following equalities hold:

$$\begin{aligned}\mathbf{\Pi}_N &= \mathbf{\Pi}_N^\top = \mathbf{\Pi}_N^2, & \mathbf{Q}_N^\top \mathbf{Q}_N &= \mathbf{\Pi}_N, \\ \mathbf{L}(t) \mathbf{\Pi}_N &= \mathbf{\Pi}_N \mathbf{L}(t) = \mathbf{L}(t).\end{aligned}$$

Moreover, we have that the spectrum of the matrix

$$\bar{\mathbf{L}}(t) := \mathbf{Q}_N \mathbf{L}(t) \mathbf{Q}_N^\top \in \mathbb{R}^{(N-1) \times (N-1)}$$

is equal to the spectrum of the extended Laplacian  $\mathbf{L}(t)$  without the eigenvalue  $\lambda_1 = 0$  corresponding to the eigenvector  $\mathbf{1}_N$ . Finally, we define the *consensus error state*  $\zeta(t) := [\zeta_1^\top(t), \zeta_2^\top(t)]^\top$  as

$$\begin{aligned}\zeta_1(t) &:= \mathbf{Q}_N \mathbf{x}(t) & \in \mathbb{R}^{N-1}, \\ \zeta_2(t) &:= \chi(t) - \rho \mathbf{1}_n + \mathbf{d} & \in \mathbb{R}^n.\end{aligned}$$

Note that, by definition,  $\zeta_1(t) = \zeta_2(t) = \mathbf{0}$  is equivalent to  $\mathbf{x}(t) \in \text{span}\{\mathbf{1}_N\}$  and  $\dot{\mathbf{x}}(t) = \rho \mathbf{1}_N$ .

With the above notation, the closed-loop collective dynamics can be reformulated as (see Appendix)

$$\dot{\zeta}(t) = \mathbf{A}_\zeta(t) \zeta(t), \quad \zeta(0) = \zeta_0, \quad (8)$$

where  $\mathbf{A}_\zeta(t) \in \mathbb{R}^{(N+n-1) \times (N+n-1)}$  is given by

$$\mathbf{A}_\zeta(t) := \begin{bmatrix} -k_P \bar{\mathbf{L}}(t) & \mathbf{Q}_N \mathbf{C} \\ -k_I \mathbf{C}^\top \mathbf{Q}_N^\top \bar{\mathbf{L}}(t) & \mathbf{0} \end{bmatrix}.$$

Next we show that, if the connectivity of the graph  $\mathcal{G}_0(t)$  verifies the PE-like condition (3), then protocol (7) solves the consensus problem (2). The next theorem proves this result.

**Theorem 1:** Consider the collective dynamics (8) and suppose that  $\mathcal{G}_0(t)$  verifies the PE-like condition (3) for some parameters  $\mu$  and  $T$ . Then, for any  $k_\beta \geq 2$ , there exist coordination gains  $k_P$  and  $k_I$  such that the inequality

$$\|\zeta(t)\| \leq \alpha_\zeta \|\zeta(0)\| e^{-\lambda_c t}$$

holds for some positive constant  $\alpha_\zeta \in (0, \infty)$ , and with

$$\lambda_c \geq \bar{\lambda}_c := \frac{k_P N \mu_{n_\ell}}{(1 + k_P N T)^2} (1 + k_\beta \frac{N}{n_\ell})^{-1}.$$

Also, the coordination states and their rates of change satisfy

$$\begin{aligned}\lim_{t \rightarrow \infty} |x_i(t) - x_j(t)| &= 0, & i, j &\in \mathcal{I}_n, \\ \lim_{t \rightarrow \infty} \dot{x}_i(t) &= \rho, & i &\in \mathcal{I}_n.\end{aligned}$$

**Proof.** The proof, which is omitted here due to space limitations, is similar to the proof of Lemma 3 in [1].  $\square$

**Remark 3:** Theorem 1 above indicates that the QoS of the network (characterized by  $T$  and  $\mu$ ) limits the achievable (guaranteed) rate of convergence of the closed-loop collective dynamics. According to the theorem, for a given QoS of the

network, the maximum (guaranteed) rate of convergence  $\bar{\lambda}_c^*$  is achieved by setting  $k_P = \frac{1}{TN}$ , which results in

$$\bar{\lambda}_c^* := \frac{\mu_{n_\ell}}{4T} \left(1 + k_\beta \frac{N}{n_\ell}\right)^{-1}.$$

We also note that, as  $T$  goes to zero (graph connected pointwise in time), the convergence rate can be set arbitrarily fast by increasing the coordination gains  $k_P$  and  $k_I$ .

### IV. CONVERGENCE UNDER QUANTIZATION

In this section we analyze the stability and performance characteristics of the distributed PI protocol presented in the previous section when the agents exchange quantized measurements. For the sake of simplicity, in this paper we consider only *uniform quantizers* with step size  $\Delta$ .

#### A. Protocol and Collective Dynamics

When only quantized information from the other agents is available, the PI protocol introduced in (7) becomes

$$\begin{aligned}\mathbf{u} &= -k_P \left( \tilde{\mathbf{D}}(t) \mathbf{x} - \tilde{\mathbf{A}}(t) \mathbf{q}(\mathbf{x}) \right) + \begin{bmatrix} \rho \mathbf{1}_{n_\ell} \\ \chi \end{bmatrix}, \\ \dot{\chi} &= -k_I \mathbf{C}^\top \left( \tilde{\mathbf{D}}(t) \mathbf{x} - \tilde{\mathbf{A}}(t) \mathbf{q}(\mathbf{x}) \right), & \chi(0) &= \chi_0,\end{aligned} \quad (9)$$

where  $\mathbf{q}(\mathbf{x}(t)) \in \mathbb{Z}^N \Delta$  is the quantized coordination state

$$\mathbf{q}(\mathbf{x}(t)) := [\mathbf{q}_\Delta(x_{\ell 1}(t)), \dots, \mathbf{q}_\Delta(x_{n_\ell}(t))]^\top,$$

with  $\mathbf{q}_\Delta(\cdot) : \mathbb{R} \rightarrow \mathbb{Z} \Delta$  being defined as

$$\mathbf{q}_\Delta(\xi) := \text{sgn}(\xi) \Delta \left\lfloor \frac{|\xi|}{\Delta} + \frac{1}{2} \right\rfloor, \quad \xi \in \mathbb{R}.$$

The time-varying matrices  $\tilde{\mathbf{D}}(t)$  and  $\tilde{\mathbf{A}}(t)$  are defined as

$$\tilde{\mathbf{D}}(t) := \mathbf{D}(t) + \mathbf{D}_\ell, \quad \tilde{\mathbf{A}}(t) := \mathbf{A}(t) + \mathbf{D}_\ell,$$

where  $\mathbf{D}(t)$  and  $\mathbf{A}(t)$  are respectively the *degree* and *adjacency matrices* of  $\mathbf{L}(t)$ , while  $\mathbf{D}_\ell$  is given by

$$\mathbf{D}_\ell := \begin{bmatrix} \mathbf{0} & -\mathbf{I}_{n_\ell} & \mathbf{0} \\ -\mathbf{I}_{n_\ell} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{N \times N}.$$

Note that only the information exchanged over the network is subject to quantization; in fact, each agent has access to its own unquantized state, and leaders also have access to the unquantized state of its virtual agent (and viceversa).

Then, noting that  $\mathbf{L}(t) = \tilde{\mathbf{D}}(t) - \tilde{\mathbf{A}}(t)$ , the collective dynamics can be written as

$$\dot{\mathbf{x}} = -k_P \mathbf{L}(t) \mathbf{x} + \begin{bmatrix} \rho \mathbf{1}_{n_\ell} \\ \chi + \mathbf{d} \end{bmatrix} + k_P \tilde{\mathbf{A}}(t) \mathbf{e}_x, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

$$\dot{\chi} = -k_I \mathbf{C}^\top \mathbf{L}(t) \mathbf{x} + k_I \mathbf{C}^\top \tilde{\mathbf{A}}(t) \mathbf{e}_x, \quad \chi(0) = \chi_0,$$

where  $\mathbf{e}_x(t) := \mathbf{q}(\mathbf{x}(t)) - \mathbf{x}(t)$  is the *quantization error vector*. In terms of the consensus error state  $\zeta(t)$ , the collective dynamics can be expressed as

$$\dot{\zeta}(t) = \mathbf{A}_\zeta(t) \zeta(t) + \mathbf{B}_\zeta(t) \mathbf{e}_x(t), \quad \zeta(0) = \zeta_0, \quad (10)$$

where  $\mathbf{A}_\zeta(t)$  was introduced in (8) and  $\mathbf{B}_\zeta(t)$  is given by

$$\mathbf{B}_\zeta(t) := \begin{bmatrix} k_P \mathbf{Q}_N \tilde{\mathbf{A}}(t) \\ k_I \mathbf{C}^\top \tilde{\mathbf{A}}(t) \end{bmatrix}.$$

Note that, in this case, the right-hand side of the collective dynamics is discontinuous not only due to the time-varying topology, but also due to the presence of quantized states. As proven in [8], Carathéodory solutions might not exist for quantized consensus problems, implying that a weaker concept of solution has to be considered. Similar to [8], we will consider solutions *in the sense of Krasovskiy* [17].



To show that Krasovsky solutions to (10) exist (at least) locally, we note that, during continuous evolution of the system between “quantization jumps”, the network dynamics (10) are linear, with the quantized state  $q(x(t))$  acting as a bounded exogenous input. This implies that the solutions  $x(t)$  are locally bounded (no *finite escape time* occurs). Then, local existence of Krasovsky solutions is guaranteed by the fact that the right-hand side of (10) is measurable and locally bounded [17]. At this point, we cannot claim that Krasovsky solutions to (10) are complete; for this, we will need to prove that solutions are bounded (see Theorem 2).

### B. (Krasovsky) Equilibria

Before investigating the convergence properties of the quantized collective dynamics (10), in this section we analyze the existence of equilibria for these dynamics. To simplify the analysis, we assume (only in this section) that the network topology is static and connected. Under this assumption, one can easily show that the unquantized collective dynamics (8) have one isolated equilibrium point at  $\zeta_{\text{eq}} = \mathbf{0}$ . However, when quantized information is exchanged over the network,  $\zeta_{\text{eq}} = \mathbf{0}$  is not an equilibrium point of the collective dynamics anymore and other (undesirable) equilibria might exist, depending on the step size of the quantizers.

To show this, we first notice that  $\dot{\zeta}(t) \equiv \mathbf{0}$  is equivalent to  $\dot{x}(t) \in \text{span}\{\mathbf{1}_N\}$  and  $\dot{\chi}(t) \equiv \mathbf{0}$  holding simultaneously. Hence,  $\zeta_{\text{eq}} := [\zeta_{1\text{eq}}^\top, \zeta_{2\text{eq}}^\top]^\top$  is an equilibrium of (8) if

$$\gamma(t)\mathbf{1}_N \in \mathcal{K} \left( -k_P \left( \tilde{D}x_{\text{eq}}(t) - \tilde{A}q(x_{\text{eq}}(t)) \right) + \begin{bmatrix} \rho \mathbf{1}_{n_\ell} \\ \chi_{\text{eq}} + d \end{bmatrix} \right), \\ \mathbf{0} \in \mathcal{K} \left( -k_I C^\top \left( \tilde{D}x_{\text{eq}}(t) - \tilde{A}q(x_{\text{eq}}(t)) \right) \right),$$

where  $\gamma(t) \in \mathbb{R}$  is an arbitrary signal;  $x_{\text{eq}}(t)$  is a continuous coordination-state trajectory satisfying  $\zeta_{1\text{eq}} = Q_N x_{\text{eq}}(t)$ ; while  $\chi_{\text{eq}} := \zeta_{2\text{eq}} - \rho \mathbf{1}_n + d$ . The second inclusion above and continuity of  $x_{\text{eq}}(t)$ , along with the fact that the network is assumed to be static and connected, preclude the existence of equilibria involving time-varying coordination-state trajectories, i.e.  $\gamma(t) \equiv 0$  (or equivalently  $\dot{x}_{\text{eq}}(t) \equiv 0$ ). Then, the set of (Krasovsky) equilibria of (8) can be defined as

$$E := \left\{ (x_{\text{eq}}, \chi_{\text{eq}}) \in \mathbb{R}^N \times \mathbb{R}^n : \right. \\ \left. \mathbf{0} \in \mathcal{K} \left( \begin{bmatrix} -k_P (\tilde{D}x_{\text{eq}} - \tilde{A}q(x_{\text{eq}})) + \begin{bmatrix} \rho \mathbf{1}_{n_\ell} \\ \chi_{\text{eq}} + d \end{bmatrix} \\ -k_I C^\top (\tilde{D}x_{\text{eq}} - \tilde{A}q(x_{\text{eq}})) \end{bmatrix} \right) \right\}. \quad (11)$$

Next, we show that, under sufficiently fine quantization, the set  $E$  is empty.

**Lemma 2:** Consider the quantized collective dynamics (10), and assume the network topology is static and connected. If the step size of the quantizers satisfies

$$\Delta < \frac{2n_\ell}{n(n-1)} \frac{|\rho|}{k_p}, \quad (12)$$

then the set of equilibria  $E$  is empty.

**Proof.** The proof is omitted due to space limitations.  $\square$

### C. Convergence Analysis

Next we show that, if the connectivity of  $\mathcal{G}_0(t)$  verifies the PE-like condition (3), then protocol (9) solves the consensus problem (2) in a practical sense. Moreover, the consensus

error state degrades gracefully with the value of the quantizer step size. The next theorem summarizes this result.

**Theorem 2:** Consider the closed-loop collective dynamics (10) and suppose that the topology  $\mathcal{G}_0(t)$  verifies the PE-like condition (3) for some parameters  $\mu$  and  $T$ . Then, there exist coordination gains  $k_P$  and  $k_I$  ensuring that there is a finite time  $T_b \geq 0$  such that the bounds

$$|x_i(t) - x_j(t)| \leq \alpha_\eta \Delta, \\ |\dot{x}_i(t) - \rho| \leq \alpha_\rho \Delta,$$

hold for all  $t \geq T_b$  and some constants  $\alpha_\eta, \alpha_\rho \in (0, \infty)$ .

**Proof.** The proof is omitted due to space limitations.  $\square$

## V. SIMULATION RESULTS

We now present simulation results illustrating the theoretical findings of the paper. To this end, we consider a network of 5 agents with dynamics (1). At a given time  $t$ , the information flow is characterized by one of the graphs in Figure 2; note that all four graphs are *not* connected. The control objective is to design a distributed PI protocol that solves the consensus problem (2) with  $\rho = 1$  (in a practical sense). In all of the simulations, the initial coordination-state vector  $x_0$  and the disturbance vector  $d$  are given by

$$x_0 = [-1, 2, 4, -4, 3]^\top, \quad d = [0, 5, -3, 4, 1]^\top.$$

To solve the consensus problem, we add 2 virtual agents to the network, and implement the (quantized) protocol (9) with PI gains  $k_P = 0.60$  and  $k_I = 0.15$ , and initial integrator state  $\chi_0 = \mathbf{0}$ . Figure 3 presents the computed evolution of the closed-loop collective dynamics with quantizer step size  $\Delta = 0.3$  (note that this step size verifies inequality (12)). The figure shows the time evolution of the coordination states, their time-derivative, the integrator states, and the 2-norms of the consensus error states  $\zeta_1(t)$  and  $\zeta_2(t)$ . Additionally, Figure 4 shows an estimate of the QoS of both the original network and the extended network, computed as

$$\hat{\mu}(t) := \lambda_{\min} \left( \frac{1}{n} \frac{1}{T} \int_{t-T}^t Q_N L_0(\tau) Q_N^\top d\tau \right), \quad t \geq T, \\ \hat{\mu}_{n_\ell}(t) := \lambda_{\min} \left( \frac{1}{N} \frac{1}{T} \int_{t-T}^t Q_N L(\tau) Q_N^\top d\tau \right), \quad t \geq T,$$

with  $T = 0.25$  sec. The results demonstrate that the PI distributed protocol allows the followers to ‘learn’ the reference rate command  $\rho$  and reach agreement with the leaders, while effectively compensating for the (constant) disturbances present in the network. To illustrate the effect of

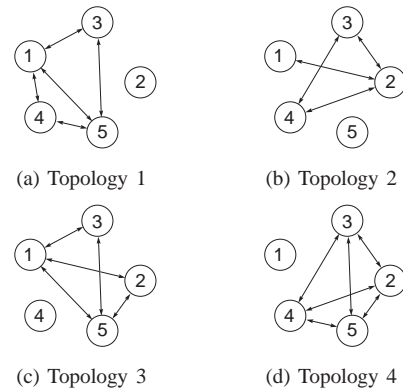
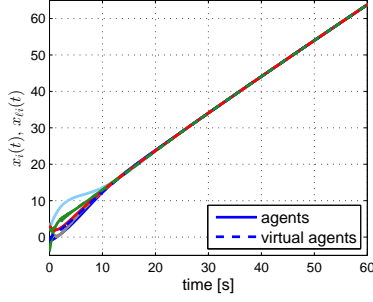
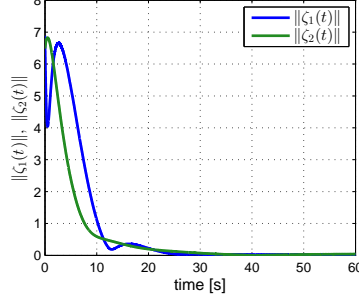


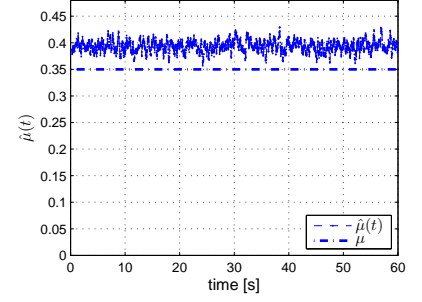
Fig. 2. Network topologies.



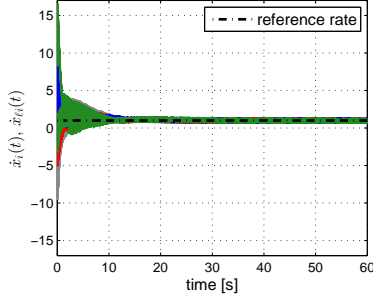
(a) Coordination states,  $x(t)$



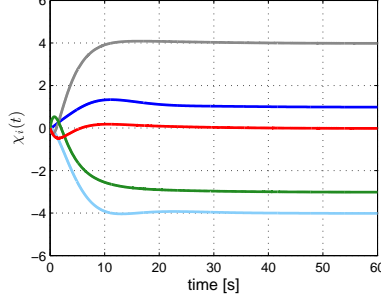
(b) Consensus error states,  $\|\zeta_i(t)\|$



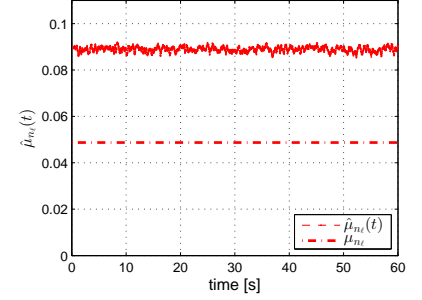
(a) Original network



(c) Rate of the coordination states,  $\dot{x}(t)$



(d) Integrator states,  $\chi(t)$



(b) Extended network

Fig. 3. Quantized closed-loop collective dynamics with fine quantization ( $\Delta = 0.3$ ).

Fig. 4. Quality of service ( $T = 0.25$  sec).

the QoS on the convergence rate of the collective dynamics, Figure 5 presents convergence times<sup>1</sup> (normalized to the case of 2 leaders and complete graph as network topology) as a function of the parameter  $\mu$  (with  $T = 0.25$  sec). The figure shows that the speed of convergence decreases with (i) the QoS of the network and (ii) the addition of virtual agents. The latter can be explained by the reduction of the QoS of the extended network as virtual agents are added (see Lemma 1). Notice that these results are consistent with Theorem 1.

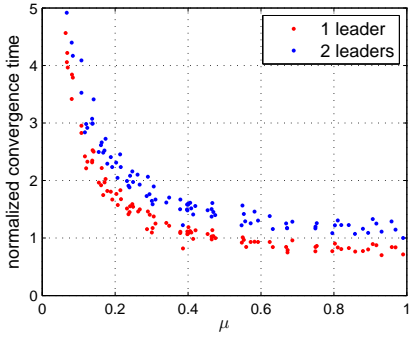


Fig. 5. Convergence time of the collective dynamics (normalized to the case of 2 leaders and complete graph as network topology;  $t_{\text{norm}} = 28.4$  sec).

Next, we use the same simulation scenario to illustrate the importance of adding virtual agents to the network. For this purpose, we consider the network of 5 agents above and use protocol (4) to drive agents 1 and 2, while agents 3, 4, and 5 are driven with protocol (5)-(6). The key difference with the simulation in Figure 3 is thus that protocol (4) is applied directly to the (uncertain) agents, rather than to

the corresponding (disturbance-free) virtual agents, which are not implemented in this case. Figure 6 shows the evolution of the collective dynamics under the same information flow (see Figure 4) and with quantizer step size  $\Delta = 0.3$ . As can be seen, the agents do not reach agreement and the coordination states do not evolve at the desired reference rate  $\rho = 1$ .

The same network of 5 agents and information flow are now used to verify that the multi-leader PI protocol (with virtual agents) is robust to the loss of a leader. In this case, we simulate the sudden loss of agent 2, which is one of the leaders. Figure 7 presents the response of the collective dynamics, which shows that, despite the loss of one of the leaders, the PI protocol is still able to solve the coordination problem (2). We note that, for the agents to reach the desired agreement, it is required that the resulting information flow still satisfy a PE-like condition similar to (3).

Finally, the same scenario is used to illustrate the existence of undesirable attractors in the presence of coarse quantization. For this purpose, we change the quantizer step size to  $\Delta = 3$ . The computed response of the closed-loop collective dynamics is shown in Figure 8. In this case, the agents do not reach the desired agreement and, in fact, the solution converges to a neighborhood<sup>2</sup> of one of the (Krasovskiy) equilibrium points characterized by (11).

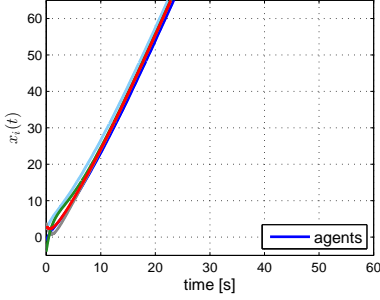
## VI. CONCLUSIONS

In this paper we analyzed the convergence properties of a distributed PI protocol to coordinate a network of agents subject to constant disturbances. We addressed the situation where each agent transmits only its coordination state to only

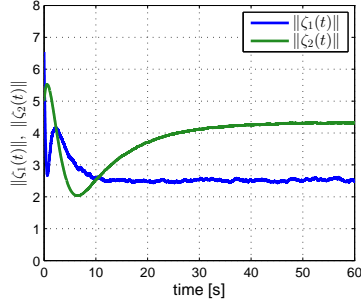
<sup>1</sup>Convergence time is defined here as the time it takes for the 2-norm of the consensus error state  $\zeta(t)$  to converge to a 2%-tube of its initial value.

<sup>2</sup>Notice that the results in Section IV-B are derived for network topologies that are both static and connected; instead, the simulations presented here consider a time-varying information flow.



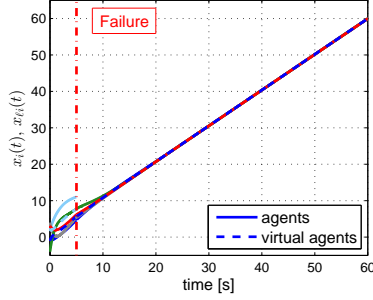


(a) Coordination states,  $\mathbf{x}(t)$

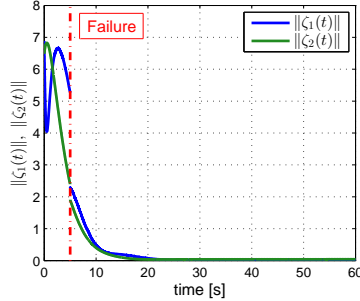


(b) Consensus error states,  $\|\zeta_i(t)\|$

Fig. 6. Collective closed-loop dynamics with no virtual agents ( $\Delta = 0.3$ ).

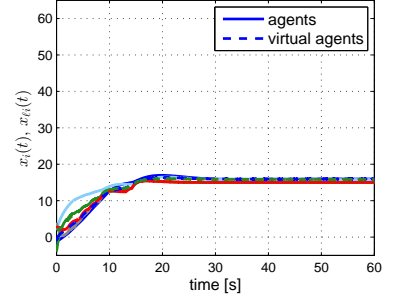


(a) Coordination states,  $\mathbf{x}(t)$

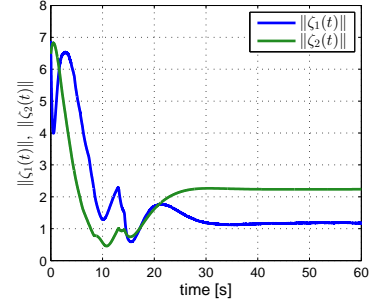


(b) Consensus error states,  $\|\zeta_i(t)\|$

Fig. 7. Robustness of the protocol to the loss of a leader ( $\Delta = 0.3$ ).



(a) Coordination states,  $\mathbf{x}(t)$



(b) Consensus error states,  $\|\zeta_i(t)\|$

Fig. 8. Collective closed-loop dynamics with coarse quantization ( $\Delta = 3$ ).

a subset of the other agents, as determined by the network topology. Furthermore, we considered the case where the graph that captures the information flow is not connected during some interval of time or even fails to be connected at all times. We also analyzed the convergence properties of the protocol when the agents exchange quantized measurements.

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## APPENDIX

### CLOSED-LOOP COLLECTIVE DYNAMICS

From the definition of  $\zeta_1(t)$  and  $\zeta_2(t)$  and the coordination-state dynamics, it follows that

$$\begin{aligned}\dot{\zeta}_1(t) &= -k_P Q_N L(t) \mathbf{x}(t) + \rho Q_N \mathbf{1}_N + Q_N [\zeta_2^0] \\ &= -k_P Q_N L(t) \mathbf{x}(t) + Q_N C \zeta_2(t).\end{aligned}$$

The properties of the projection matrix  $\Pi_N$ , along with the fact that  $Q_N Q_N^\top = \mathbb{I}_{N-1}$ , imply that

$$\begin{aligned}\dot{\zeta}_1(t) &= -k_P Q_N \Pi_N L(t) \Pi_N \mathbf{x}(t) + Q_N C \zeta_2(t) \\ &= -k_P Q_N Q_N^\top Q_N L(t) Q_N^\top Q_N \mathbf{x}(t) + Q_N C \zeta_2(t) \quad (13) \\ &= -k_P \bar{L}(t) \zeta_1(t) + Q_N C \zeta_2(t).\end{aligned}$$

Similarly, it follows that

$$\dot{\zeta}_2(t) = -k_I C^\top L(t) \mathbf{x}(t) = -k_I C^\top Q_N^\top \bar{L}(t) \zeta_1(t). \quad (14)$$

Equations (13) and (14) lead to the dynamics (8).